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The Solution by Iteration of Nonlinear Equations in Uniformly Smooth Banach Spaces

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Let E be a uniformly smooth Banach space and let $T: D(T) \subset E \mapsto E$ be a strong pseudocontraction with an open domain $D(T)$ in E and a fixed point $x^* \in D(T)$. We establish the strong convergence of the Mann and Ishikawa iterative processes (with errors) to the fixed point of T. Related results deal with the iterative solution of operator equations of the forms $f \in Tx$ and $f \in x + \lambda Tx$, $\lambda > 0$, when T is a set-valued strongly accretative operator. Our theorems include the cases in which the operator T is defined only locally. Explicit error estimates are also given. © 1997 Academic Press

1. INTRODUCTION

Let E be a normed linear space. A mapping $T: D(T) \subset E \mapsto E$ is said to be *strongly pseudocontractive* if there exists $t > 1$ such that for all $x, y \in D(T)$ and $r > 0$

$$\|x - y\| \leq \|(1 + r)(x - y) + rt(Tx - Ty)\|. \tag{1}$$

It $t = 1$, then T is said to be *pseudocontractive*. This class of operators is of interest mainly because of its intimate connections with the important class of nonlinear *accretive operators* where a mapping U with domain $D(U)$ and range $R(U)$ in E is said to be *accretive* if for all $x, y \in D(T)$ and all $s > 0$, we have that

$$\|x - y\| \leq \|x - y + s(Ux - Uy)\|. \tag{2}$$

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From inequalities (1) and (2) it is clear that T is (strongly) pseudocontractive if and only if $U = (I - T)$ is (strongly) accretive. Thus, the mapping theory of accretive operators is closely related to the fixed point theory of pseudocontractive operators.

Interest in the accretive operators (which were introduced by Browder [2] and Kato [21] independently in 1967) stems mainly from their connection with the solvability of evolution equations. An early fundamental result of Browder [2] states that the initial-value problem

$$\frac{du}{dt} + Tu = 0; \qquad u(0) = u_0, \tag{3}$$

is solvable if T is locally Lipschitz and accretive. Consequently, pseudocontractive and accretive operators have been studied extensively by various authors (see, e.g., [1–23, 25–35]). We observe that if $N(T)$ denotes the kernel of T, then the members of $N(T)$ are, in fact, the equilibrium points of the system (3). Consequently, considerable effort has been devoted to developing constructive techniques for the determination of the kernels of accretive operators (or the fixed points of pseudocontractions) (see, e.g., [1, 4–20, 22–32, 35]).

Recently, Chidume [5] established the strong convergence of an iteration process of *Mann type* (see, e.g., [24]) to the fixed point of a *Lipschitz* strong pseudocontraction which maps a bounded closed convex nonempty subset of L_p (or l_p), $p \geq 2$ into itself. This result has been extended by many authors in various directions (see, e.g., [6, 8, 13, 16–18, 22, 29, 30, 32]). Recently, J. Schu [29] extended the result to an iteration process of the *Ishikawa type* and to a more general class of Banach spaces; the so-called *Banach spaces with property* $(U, \lambda, m+1, m)$ (see, e.g., [20, 29]). In [30] he established the strong convergence of the Mann process to a fixed point of T when T is a *uniformly continuous* strong pseudocontraction and maps a bounded closed convex and nonempty subset of a real *smooth Banach space* into itself. Chidume [10] proved the same result for the Ishikawa iteration scheme. This result has recently been extended to *arbitrary real Banach spaces* by Chidume and Osilike in [14].

In several applications, it is well known that the operator T of (1) need not be defined on the whole of E. If the domain of T, $D(T)$, is a *proper* subset of the Banach space E and T maps $D(T)$ into E, then neither the Mann nor the Ishikawa iteration sequence may be well defined. Several authors have used the Mann and the Ishikawa iteration methods for this problem (see our *General remarks* (6) below). Very recently, Chidume [8] proved the following theorems.

THEOREM C1 [8]. *Let E be a real q-uniformly smooth Banach space. Suppose T is a Lipschitz strongly pseudocontractive map with open domain $D(T)$ in E. Suppose further that T has a fixed point $x^* \in D(T)$. Let $c_n = 1/k(n+\mu)$ where $\mu = [L^q c/k^q]^{1/(q-1)}$. Then for any $R > 0$ such that $B = B_R(x^*) = \{x \in E: \|x - x^*\| \leq R\}$ is contained in $D(T)$ and for any initial guess $x_1 \in B$, the sequence $\{x_n\}_{n=1}^{\infty}$ defined by*

$$x_{n=1} = (1 - c_n)x_n + c_n T x_n; \qquad n \geq 1,$$

remains in $D(T)$ and converges strongly to x^* with

$$\|x_n - x^*\| = 0(n^{-(q-1)/q}).$$

THEOREM C2 [8]. *Let E, K, R, B, and T be as in Theorem* C1. *Let $c_n = 1/k^2(n + \mu^*)$, $\beta_n = k(1-k)/(n+M)$. Then for arbitrary $x_1 \in B$, the sequence $\{x_n\}_{n=1}^{\infty}$ generated from x_1 by*

$$x_{n+1} = (1 - c_n)x_n + c_n T y_n$$
$$y_n = (1 - \beta_n)x_n + \beta_n T x_n, \qquad n \geq 1$$

remains in $D(T)$ and converges to x^* with $\|x_n - x^*\| = 0(n^{-(q-1)/q})$.

THEOREM C3 [8]. *Suppose T is a set-valued accretive operator with open domain $D(T)$ in E and $f \in Tx$ has a solution $x^* \in D(T)$. Then there exist a neighbourhood $B \subseteq D(T)$ of x^* and a real number $r_1 > 0$ such that for any $r > r_1$, any initial guess $x_1 \in B$, any single-valued selection T_0 of T, and some real sequence $\{c_n\}_{n=1}^{\infty}$ with $c_n = 1/k(n+r)$, the sequence $\{x_n\}_{n=1}^{\infty}$ generated from x_1 by*

$$x_{n+1} = x_n + c_n(f - T_0 x_n + x_n), \qquad n \geq 1$$

remains in $D(T)$ and converges strongly to x^* with $\|x_n - x^*\| = 0(n^{-(q-1)/q})$.

It is our purpose in this paper to extend these results of Chidume [8] to the more general *uniformly smooth Banach spaces* and *without any continuity assumption whatsoever* on T. As a consequence of our theorems, we shall obtain related results on the iterative solution of nonlinear operator equations of the form $f \in Tx$ or $f \in x + \lambda Tx$, $\lambda > 0$, involving set-valued *m-accretive* and *m-dissipative* operators. We shall also discuss the special cases when the operator T is defined only locally. Finally, we shall indicate how our theorems easily extend to iterative schemes with errors. Our theorems extend and unify most of the results that have recently appeared. In particular, Theorems 3.1, 3.2, 3.4, and 4.1 of [8]; Theorems 3 and 5 of [13]; Theorems 7 and 11 of [11]; Theorem 1 of [19]; Theorems 1 and 2 of [23], Theorem 1 of [32], and a host of other theorems will be special cases of our theorems.

2. PRELIMINARIES

Let E be a real normed linear space with $\dim \cdot E \geq 2$. The *modulus of smoothness* of E is defined by

$$\rho_E(\tau) = \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\}; \qquad \tau > 0. \quad (4)$$

Let $p > 1$ be a real number. E is said to be *p-uniformly smooth* if there exist a constant $c > 0$ such that

$$\rho_E(\tau) \leq c\tau^p. \quad (5)$$

Prototypes of the p-uniformly smooth spaces are the L_p, l_p, W_p^m, $1 < p < \infty$, spaces. E is said to be *uniformly smooth* if

$$\lim_{\tau \mapsto 0} \frac{\rho_E(\tau)}{\tau} = 0. \quad (6)$$

Let E^* be the dual space of E and let $\langle \cdot , \cdot \rangle$ denote the generalised duality pairing between elements of E and elements of E^*. Then we denote by $J: E \mapsto 2^{E^*}$ the normalised duality mapping defined by

$$Jx = \left\{f^* \in E^*: \langle x, f^*\rangle = \|x\|^2; \|f^*\| = \|x\|\right\}.$$

It is known that if E is uniformly smooth then J is single-valued and uniformly continuous on bounded subsets of E. We denote this single-valued duality map by j.

The following results will be needed in the sequel.

LEMMA X-R [34]. *Let E be a real uniformly smooth Banach space. Then for every $x, y \in E$ and some positive constants D and C*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + D \max\left\{\|x\| + \|y\|, \frac{C}{2}\right\}\rho_E(\|y\|). \quad (7)$$

LEMMA W [32]. *Let $\{\Psi_n\}_{n \geq 0}$ be a nonnegative sequence of real numbers such that*

$$\Psi_{n+1} \leq (1 - \delta_n)\Psi_n + \sigma_n, \quad (8)$$

where $\delta_n \in [0, 1]$, $\sum_{n \geq 0} \delta_n = \infty$, $\sigma_n = o(\delta_n)$. Then $\Psi_n \to 0$ as $n \to \infty$.

3. MAIN RESULTS

3.1. *Convergence Theorems for Strong Pseudocontractions*

THEOREM 1. *Let E be a uniformly smooth Banach space and let $T: D(T) \mapsto E$ be a strong pseudocontraction with an open domain $D(T) \subset E$. Suppose T has a fixed point $x^* \in D(T)$. Then there exist a real number $\mu \geq 1$ and a neighbourhood B of x^* such that starting with an arbitrary $x_0 \in B$, the sequence $\{x_n\}_{n \geq 0}$ defined iteratively by*

$$y_n = (1 - \beta_n)x_n + \beta_n Tx_n; \qquad n \geq 0 \tag{9}$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n; \qquad n \geq 0 \tag{10}$$

remains in B and converges strongly to x^* with $\|x_n - x^*\| = 0(n^{-1/2})$ where $\alpha_n = 1/2(n + \mu) = \beta_n \; \forall n \geq 0$.

Proof. Observe that $(I - T)$ is strongly accretive and so is locally bounded at each interior point of its effective domain. Since $D(T)$ is open, we can choose $r > 0$ such that $B = \overline{B}_r(x^*) = \{x \in E: \|x - x^*\| \leq r\}$ is contained in $D(T)$ and $(I - T)(B)$ is bounded. Let $d = \operatorname{diam} \cdot (I - T)(B)$ and set $M = D\max\{d + r, C/2\}$; $d^* = 2d$; $A = D\max\{d^* + r, C/2\}$ where D and C are the constants appearing in (7). Since E is uniformly smooth, $\tau^{-1}\rho_E(d^*\tau) \to 0$ as $\tau \to 0$. Thus, we can choose $\tau_0 > 0$ such that $\forall 0 < \tau \leq \tau_0$, $\tau^{-1}\rho_E(d^*\tau) \leq \min\{kr^2/3A, kr^2/M\}$. By the uniform continuity of j on bounded sets, given $0 < \varepsilon = kr^2/6d$ we can choose $\delta_\varepsilon > 0$ such that $\|x - y\| \leq \delta_\varepsilon \Rightarrow \|j(x) - j(y)\| \leq \varepsilon$. Choose any $0 < \delta_0 < \delta_\varepsilon$. Define $\mu := \max\{1, 1/2\tau_0, d/2\delta_0, 6d/r\}$ and generate the sequence $\{x_n\}_{n \geq 0}$ iteratively by (9) and (10).

We now show that this sequence is well-defined and is in B. To do this, we first prove that $y_n \in B$ whenever $x_n \in B$. Let $x_n \in B$. Then,

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|x_n - x^* - \beta_n(x_n - Tx_n)\|^2 \\
&\leq \|x_n - x^*\|^2 - 2\beta_n\langle x_n - Tx_n, j(x_n - x^*)\rangle \\
&\quad + D\max\left\{\|x_n - x^*\| + \beta_n\|x_n - Tx_n\|, \frac{C}{2}\right\} \\
&\quad \times \rho_E(\beta_n\|x_n - Tx_n\|) \\
&\leq (1 - 2k\beta_n)\|x_n - x^*\|^2 + M\beta_n\left[\frac{\rho_E(d\beta_n)}{\beta_n}\right] \leq r^2.
\end{aligned}$$

It now suffices to prove that $x_n \in B \; \forall n \geq 0$. We proceed by induction. By our choice, $x_0 \in B$. Assume $x_n \in B$. Then, by the argument above, $y_n \in B$.

Moreover,

$$\begin{aligned}
\| x_{n+1} - x^* \|^2 &= \| x_n - x^* - \alpha_n (x_n - Ty_n) \|^2 \\
&\le \| x_n - x^* \|^2 - 2\alpha_n \langle x_n - Ty_n , j(x_n - x^*) \rangle \\
&\quad + D \max\left\{ \| x_n - x^* \| + \alpha_n \| x_n - Ty_n \|, \frac{C}{2} \right\} \\
&\quad \times \rho_E (\alpha_n \| x_n - Ty_n \|)
\end{aligned}$$

$$\begin{aligned}
\langle x_n - Ty_n , j(x_n - x^*) \rangle &= \langle x_n - y_n , j(x_n - x^*) \rangle + \langle y_n - Ty_n , j(x_n - x^*) \rangle \\
&= \beta_n \langle x_n - Tx_n , j(x_n - x^*) \rangle \\
&\quad + \langle y_n - Ty_n , j(y_n - x^*) \rangle \\
&\quad - \langle y_n - Ty_n , j(y_n - x^*) - j(x_n - x^*) \rangle \\
&\ge k\beta_n \| x_n - x^* \|^2 + k \| y_n - x^* \|^2 \\
&\quad - \| y_n - Ty_n \| \cdot \| j(y_n - x^*) - j(x_n - x^*) \|
\end{aligned}$$

$$\| y_n - x_n \| = \beta_n \| x_n - Tx_n \| \le d\beta_n = \frac{d}{2(n + \mu)} \le \frac{d}{2\mu} < \delta_0$$

so

$$\| j(y_n - x^*) - j(x_n - x^*) \| \le \frac{kr^2}{6d} .$$

Thus, we have the estimates

$$\begin{aligned}
\| y_n - x^* \|^2 &\ge \beta_n^2 \| x_n - Tx_n \|^2 + \| x_n - x^* \|^2 \\
&\quad - 2\beta_n \| x_n - Tx_n \| \cdot \| x_n - x^* \| \\
&\quad \times \langle x_n - Ty_n , j(x_n - x^*) \rangle \\
&\ge k(1 + \beta_n) \| x_n - x^* \|^2 - 2k\,dr\,\beta_n \\
&\quad - d \| j(y_n - x^*) - j(x_n - x^*) \|
\end{aligned}$$

$$\| x_n - Ty_n \| \le \beta_n \| x_n - Tx_n \| + \| y_n - Ty_n \| \le d^* .$$

We now have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le (1 - 2k\alpha_n)\|x_n - x^*\|^2 \\ &\quad + 2d\alpha_n \|j(y_n - x^*) - j(x_n - x^*)\| \\ &\quad + 4kdr\alpha_n \beta_n + A\alpha_n\left[\frac{\rho_E(d^*\alpha_n)}{\alpha_n}\right] \\ &\le (1 - k\alpha_n)r^2 \le r^2. \end{aligned} \tag{11}$$

So, $x_n \in B \quad \forall n \ge 0$. Now set $\delta_n = 2k\alpha_n$; $\Psi_n = \|x_n - x^*\|^2$; $\sigma_n = 4kdr\alpha_n \beta_n + 2d\alpha_n \|j(y_n - x^*) - j(x_n - x^*)\| + A\alpha_n[\rho_E(d^*\alpha_n)/\alpha_n]$ so that (11) now becomes

$$\Psi_{n+1} = (1 - \delta_n)\Psi_n + \sigma_n; \qquad n \ge 0. \tag{12}$$

Since $\delta_n \in [0,1]$, $\sum_{n \ge 0} \delta_n = \infty$, $\Psi_n \ge 0$, and $\sigma_n = o(\delta_n)$ we obtain $\Psi_n \to 0$ so that $x_n \to x^*$ strongly as $n \to \infty$. Moreover, $\|x_n - x^*\| = 0(n^{-1/2})$. This completes the proof.

Remark 1. The choice of $\alpha_n = 1/2(n + \mu) = \beta_n$ for the real sequences is not crucial. In particular, the theorem still holds if $\{\alpha_n\}$, $\{\beta_n\}$ are chosen satisfying the conditions

(i) $0 < \alpha_n \le \min\{1, \tau_0\}$; (ii) $0 < \beta_n \le \min\left\{1, \tau_0, \dfrac{\delta_0}{d}, \dfrac{r}{12d}\right\}$

(iii) $\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n$ (iv) $\sum_{n \ge 0} \alpha_n = \infty$.

THEOREM 2. *Let E, T be as in Theorem 1. Then there exist $\alpha \ge 1$ and a neighbourhood B of x^* such that starting with an arbitrary $x_0 \in B$, the sequence $\{x_n\}_{n \ge 0}$ defined iteratively by*

$$x_{n+1} = (1 - C_n)x_n + C_n Tx_n; \qquad n \ge 0 \tag{13}$$

remains in B and converges strongly to x^* with $\|x_n - x^*\| = 0(n^{-1/2})$ where $C_n = 1/2(n + \alpha)$ for all $n \ge 0$.

Proof. Proceeding as in the proof of Theorem 1, we set $\alpha = 1/2\tau_0$, and for arbitrary $x_0 \in B$ generate $\{x_n\}$ as in (13). We now prove that

$x_n \in B\ \forall n \geq 0$. We proceed by induction. Assume $x_n \in B$. Then,

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq (1 - 2kC_n)\|x_n - x^*\|^2 \\ &\quad + D\max\left\{\|x_n - x^*\| + C_n\|x_n - Tx_n\|, \frac{C}{2}\right\} \\ &\quad \times \rho_E(C_n\|x_n - Tx_n\|) \\ &\leq (1 - 2kC_n)\|x_n - x^*\|^2 + MC_n\left[\frac{\rho_E(dC_n)}{C_n}\right] \\ &\leq (1 - kC_n)r^2 \leq r^2.\end{aligned}$$

Thus, $x_{n+1} \in B$ and so $\{x_n\}_{n \geq 0} \subset B$. By setting $\Psi_n = \|x_n - x^*\|^2$, $\delta_n = 2kC_n$, $\sigma_n = MC_n[\rho_E(dC_n)/C_n]$, we obtain (12) and the rest follows as in the proof of Theorem 1. This completes the proof.

Remark 2. Again, the choice $C_n = 2^{-1}(n + \alpha)^{-1}$ is not crucial. In particular, the theorem remains true if we choose any $\{C_n\}_{n \geq 0}$ to satisfy the following conditions

$$\text{(i) } 0 < C_n \leq \min\{1, \tau_0\}, \qquad \text{(ii) } \lim_{n \to \infty} C_n = 0, \qquad \text{(iii) } \sum_{n \geq 0} C_n = \infty.$$

3.2. *Iterative Solution of the Equation $f \in Tx$*

THEOREM 3. *Let E, $\{\alpha_n\}$, $\{\beta_n\}$ be as in Theorem* 1 *and let $T: D(T) \mapsto E$ be a set-valued strongly accretive operator with open domain $D(T) \subset E$ such that the equation $f \in Tx$ has a solution $x^* \in D(T)$. Then, there exist a real number $\mu \geq 1$ and a neighbourhood B of x^* such that starting with an arbitrary $x_0 \in B$ and any single-valued selection T_0 of T, the sequence $\{x_n\}_{n \geq 0}$ defined iteratively by*

$$y_n = x_n + \beta_n(f - T_0 x_n); \qquad n \geq 0 \tag{14}$$

$$x_{n+1} = x_n + \alpha_n(f - T_0 y_n); \qquad n \geq 0 \tag{15}$$

remains in B and converges strongly to x^* with $\|x_n - x^*\| = 0(n^{-1/2})$.

Proof. Observe that $f = T_0 x^*$. We choose $r > 0$ such that $B = B_r(x^*) \subset D(T)$ and $T_0(B)$ is bounded. Let $d = 2 \cdot \operatorname{diam} \cdot T_0(B)$ and $M = D\max\{d + r, C/2\}$. Let $\tau_0 > 0$ be such that $\forall 0 < \tau \leq \tau_0$, $\tau^{-1}\rho_E(d\tau) \leq kr^2/3M$. Choose μ as in Theorem 1 and define the iterative sequence by (14) and (15). We now prove that $\{y_n\}$, $\{x_n\}$ are in B. Proceeding as in

Theorem 1, we first prove that $x_n \in B \Rightarrow y_n \in B$. Now

$$
\begin{aligned}
\|y_n - x^*\|^2 \le \|x_n - x^*\|^2 &- 2\beta_n \langle T_0 x_n - T_0 x^*, j(x_n - x^*)\rangle \\
&+ D \max\left\{\|x_n - x^*\| + \beta_n \|T_0 x_n - T_0 x^*\|, \frac{C}{2}\right\} \\
&\times \rho_E(\beta_n \|T_0 x_n - T_0 x^*\|) \\
&\le (1 - 2k\beta_n)\|x_n - x^*\|^2 + M\beta_n \left[\frac{\rho_E(d\beta_n)}{\beta_n}\right] \le r^2.
\end{aligned}
$$

To show that $x_n \in B$, $\forall n \ge 0$, we proceed by induction. $x_0 \in B$ by our choice, so assume $x_n \in B$. Then, note that,

$$\|T_0 y_n - T_0 x^*\| \le \|T_0 y_n\| + \|T_0 x^*\| \le d;$$

and

$$\|T_0 x_n - T_0 x^*\| \le d.$$

Also,

$$
\begin{aligned}
&\langle T_0 y_n - T_0 x^*, j(x_n - x^*)\rangle \\
&\quad = \langle T_0 y_n - T_0 x^*, j(y_n - x^*)\rangle \\
&\qquad - \langle T_0 y_n - T_0 x^*, j(y_n - x^*) - j(x_n - x^*)\rangle \\
&\quad \ge k\|y_n - x^*\|^2 - \|T_0 y_n - T_0 x^*\| \cdot \|j(y_n - x^*) - j(x_n - x^*)\| \\
&\quad \ge -2kdr\beta_n + k\|x_n - x^*\|^2 - d\|j(y_n - x^*) - j(x_n - x^*)\|.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 \le \|x_n - x^*\|^2 &- 2\alpha_n \langle T_0 y_n - T_0 x^*, j(x_n - x^*)\rangle \\
&+ D \max\left\{\|x_n - x^*\| + \alpha_n \|T_0 y_n - T_0 x^*\|, \frac{C}{2}\right\} \\
&\times \rho_E(\alpha_n \|T_0 y_n - T_0 x^*\|) \\
&\le (1 - 2k\alpha_n)\|x_n - x^*\|^2 + 4kdr\alpha_n\beta_n \\
&\quad + 2d\alpha_n \|j(y_n - x^*) - j(x_n - x^*)\| + M\alpha_n \left[\frac{\rho_E(d\alpha_n)}{\alpha_n}\right] \\
&\le r^2.
\end{aligned}
$$

The rest now follows as in Theorem 1 and the proof is complete.

THEOREM 4. *Let E, T_0, T be as in Theorem 3. Then there exist a real number $\alpha \geq 1$ and a neighbourhood B of x^* such that starting with an arbitrary $x_0 \in B$, the sequence $\{x_n\}_{n \geq 0}$ defined iteratively by*

$$x_{n+1} = x_n + C_n(f - T_0 x_n); \qquad n \geq 0 \tag{16}$$

remains in B and converges strongly to x^* with $\|x_n - x^*\| = 0(n^{-1/2})$ where $C_n = 2^{-1}(n + \alpha)^{-1}$ $\forall n \geq 0$.

Proof. Proceeding as in the proof of Theorem 3, we set $\alpha = 1/2\tau_0$ and generate the sequence iteratively as in (16). The rest now follows easily and the proof is complete.

3.3. *Iterative Solution of the Equation $f \in x + \lambda Tx$*

THEOREM 5. *Let E be as in Theorem 1 and let $T: D(T) \mapsto E$ be a set-valued m-accretive operator with open domain $D(T) \subset E$. Assume that $f \in x + \lambda Tx$, $\lambda > 0$ has a solution $x^* \in D(T)$. Then there exist a real number $\mu \geq 1$ and a neighbourhood B of $x^* \in D(T)$ such that starting with an arbitrary $x_0 \in B$ and any single-valued selection T_0 of T, the iterative sequence $\{x_n\}_{n \leq 0}$ defined by*

$$y_n = x_n + \beta_n(f - x_n - \lambda T_0 x_n); \qquad n \geq 0 \tag{17}$$

$$x_{n+1} = x_n + \alpha_n(f - y_n - \lambda T_0 y_n); \qquad n \geq 0 \tag{18}$$

remains in B and converges strongly to x^* with $\|x_n - x^*\| = 0(n^{-1/2})$ where $\alpha_n = 2^{-1}(n + \mu)^{-1} = \beta_n$ $\forall n \geq 0$.

Proof. Observe that $f = x^* + \lambda T_0 x^*$. Choose $r > 0$ such that $B = B_r(x^*) \subset D(T)$ and $T_0(B)$ is bounded. Then set $d = 2 \cdot \operatorname{diam} \cdot T_0(B) + r$; $M = D \max\{d + r, C/2\}$. Choose μ as in Theorem 1. The rest now follows easily and the proof is complete.

COROLLARY 1. *Let E, T_0, T be as in Theorem 5. Then there exist a real number $\alpha \geq 1$ and a neighbourhood B of x^* such that starting with any $x_0 \in B$, the sequence $\{x_n\}_{n \geq 0}$ defined iteratively by*

$$x_{n+1} = x_n + C_n(f - x_n - \lambda T_0 x_n); \qquad n \geq 0 \tag{19}$$

remains in B and converges strongly to x^* with $\|x_n - x^*\| = 0(n^{-1/2})$ where $C_n = 2^{-1}(n + \alpha)^{-1}$; $\forall n \geq 0$.

THEOREM 6. *Let E be a real uniformly smooth Banach space and let T be a set-valued m-dissipative operator with an open domain $D(T)$ in E. Assume that $f \in x - \lambda Tx$, $\lambda > 0$ has a solution $x^* \in D(T)$. Then the conclusions of Theorem 5 remain valid where the iterative sequence is generated by*

$$y_n = x_n + \beta_n(f - x_n + \lambda T_0 x_n), \qquad n \geq 0 \tag{20}$$

$$x_{n+1} = x_n + \alpha_n(f - y_n + \lambda T_0 y_n), \qquad n \geq 0. \tag{21}$$

COROLLARY 2. *The conclusions of Corollary* 1 *remain valid if* T *and* x^* *are as in Theorem* 6 *and the iterative sequence is generated by*

$$x_{n+1} = x_n + C_n(f - x_n + \lambda T_0 x_n); \qquad n \geq 0. \tag{22}$$

3.4. *Iteration Methods with Errors*

The above results remain true if the iterative sequences are subjected to "small" perturbations by introducing the so-called error terms. We give a sketch of this for Theorem 1 only; the same process serves for the rest of our theorems.

COROLLARY 3. *In Theorem* 1, *let the iterative sequence* $\{x_n\}$ *be generated by*

$$y_n = (1 - \beta_n)x_n + \beta_n T x_n + u_n; \qquad n \geq 0 \tag{23}$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n + v_n; \qquad n \geq 0, \tag{24}$$

where $\|u_n\| = o(\beta_n)$ *and* $\|v_n\| = o(\alpha_n)$. *Then the conclusions of Theorem* 1 *remain valid*.

Proof. Set $d = r + \operatorname{diam} \cdot (I - T)(B)$; $d^* = 2(d + r)$ $M = D\max\{d + 2r, C/2\}$; $A = D\max\{d^* + r, C/2\}$. Choose $\tau_0 > 0$ such that $\forall 0 < \tau \leq \tau_0$, $\tau^{-1}\rho_E(d^*\tau) \leq \min\{kr^2/6A, kr^2/2M\}$. Also choose $\delta > 0$ such that $\|x - y\| \leq \delta \Rightarrow \|j(x) - j(y)\| \leq kr^2/12d$. Proceed as in the proof of Theorem 1. Observe that $\|u_n\| = \beta_n t_n$ and $\|v_n\| = \alpha_n s_n$ where $\{t_n\}$, $\{s_n\}$ are real sequences such as $t_n, s_n \to 0$ as $n \to \infty$. Then, one easily obtains

$$\|y_n - x^*\|^2 \leq (1 - 2k\beta_n)\|x_n - x^*\|^2 + 2r\beta_n t_n + M\beta_n\left[\frac{\rho_E(d\beta_n)}{\beta_n}\right]$$

and also

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \leq (1 - 2k\alpha_n)\|x_n - x^*\|^2 & \\ &+ 2r(1 + 2k)\alpha_n\beta_n t_n + 4krd\alpha_n\beta_n \\ &+ 2r\alpha_n s_n + 2d\alpha_n\|j(y_n - x^*) - j(x_n - x^*)\| \\ &+ A\alpha_n\left[\frac{\rho_E(d^*\alpha_n)}{\alpha_n}\right].\end{aligned}$$

The rest now follows easily and the proof is complete.

COROLLARY 4. *In Theorem* 2, *let the sequence* $\{x_n\}$ *be iteratively generated by*

$$x_{n+1} = (1 - C_n)x_n + C_n T x_n + u_n; \qquad n \geq 0, \tag{25}$$

where $\|u_n\| = o(C_n)$. *Then, the conclusions of Theorem* 2 *remain true*.

4. GENERAL REMARKS

(1) Remarks 1 and 2 regarding the choice of the real sequences apply globally to all the results in this paper. Therefore, the specific choices used in the theorems are not crucial.

(2) Our condition on the error terms, $\|u_n\| = o(\beta_n)$ or $o(C_n)$, and $\|v_n\| = o(\alpha_n)$ is weaker than the condition, $\sum \|v_n\| < \infty$, $\sum \|u_n\| < \infty$, which has been imposed by various authors (see, e.g., [17–19, 22, 23]) as can be seen from the following lemma.

LEMMA 1. *Let $f, g\colon \mathbb{N} \mapsto \mathbb{R}^+$ be sequences and suppose that* (i) $g_n \leq 1$; $\forall n \geq 0$, (ii) $g(n) \to 0$ *as* $n \to \infty$, (iii) $\sum_{n \geq 0} g(n) = \infty$. *Then*

$$\sum_{n \geq 0} f(n) < \infty \Rightarrow f = o(g) \qquad \text{as } n \to \infty.$$

The converse is false.

Proof. By conditions (i)–(iii), $g(n) \sim n^{-\alpha}$; $\alpha \in (0, 1]$. Now, $\sum_n f(n) < \infty \Rightarrow f(n) \sim n^{-(1+\varepsilon)}$, at least, for $\varepsilon > 0$. Then,

$$\frac{f(n)}{g(n)} \sim n^{-(1+\varepsilon-\alpha)} \to 0 \qquad \text{as } n \to \infty.$$

Conversely, let $g(n) = (1+n)^{-1/2}$ and $f(n) = (1+n)^{-1}$. Then, g satisfies conditions (i)–(iii) and $f = o(g)$, since $f(n)/g(n) = (1+n)^{-1/2} \to 0$ as $n \to \infty$. But $\sum_n f(n) = \infty$. This completes the proof.

(3) In Theorems 1 and 2 and Corollaries 3 and 4, it is sufficient that T be a *strong hemi-contraction*. That is, there exist $x^* \in F(T)$, the fixed point set of T and $t > 1$ such that $\forall r > 0$ and $x \in D(T)$ the following inequality holds

$$\|x - x^*\| \leq \|(1+r)(x - x^*) - rt(Tx - Tx^*)\|.$$

In Theorems 3 and 4, it is again sufficient that T be *strongly hemi-accretive* in the sense that there exists $x^* \in D(T)$ such that $f \in Tx^*$ and $\forall x \in D(T)$, $\forall r > 0$, and any single-valued selection T_0 of T, the following inequality holds

$$\|x - x^*\| \leq \|x - x^* + r(T_0 x - T_0 x^*)\|.$$

The same applies to Theorems 5 and 6 and Corollaries 1 and 2, with $x^* \in f - \lambda Tx^*$.

(4) The method of our proofs easily extends to the cases where the operator T is defined only locally.

(5) Theorems 1 and 2 generalise Theorems 3.1, 3.2, and 3.4 in [8] from p-uniformly smooth Banach spaces $(1 < p < \infty)$ to the more general uniformly smooth Banach spaces and from Lipschitz continuous operators to operators which need not be continuous. Corollaries 3 and 4 then extend these theorems to the case of iteration methods with errors. Theorem 4 extends Theorems 4.1 of Chidume [8] from p-uniformly smooth Banach spaces to uniformly smooth Banach spaces. Theorem 3 then extends Theorem 4 to Ishikawa iterates. The rest of our theorems are significant extensions and generalisations of important known results.

(6) Finally, we observe that in Theorem 1 of [32], where the author considered local strict pseudocontractions, the sequence $\{x_n\}$ is, in fact, not defined as can be easily seen from the following counterexample (see also *Math. Review*, No. 93i).

Let $E = l_2$, $K = \{x \in E: \|x\| \leq 1\}$, and define $T: K \mapsto E$ by $Tx = -4x$. It is easy to see that T is a Lipschitz strong pseudocontraction with the unique fixed point $x^* = (0, 0, 0, \dots)$. Take $x_0 = (1, 0, 0, \dots)$ and set $\alpha_n = 1/(n+2) = \beta_n$. Then $y_0 = -(3/2)x_0 \notin K$. Thus, Ty_0 cannot be computed and so x_n, $n \geq 1$, is not defined. Observe that neither the Mann nor the Ishikawa iterative process is well defined in this case.

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